Surface Integrals

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v))\,|\vec{S}_u \times \vec{S}_v|\,dA$$

where $\vec{S}(u,v)$ parameterizes S on D

Ex: Compute $\iint_S x^2$ for the unit sphere centered at the origin.

Sol: First we parameterize the surface:

$$\vec{S}(\theta, \varphi) = \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos\varphi \rangle$$

where $(\theta,\varphi) \in [0,2\pi] \times [0,\pi]$

Sphere where $\rho = 1$, nicely parameterizes surface

Find $\vec{S}_\theta = \langle -\sin(\varphi)\sin(\theta), \sin(\varphi)\cos(\theta), 0 \rangle$

$= \sin(\varphi)\langle -\sin\theta, \cos\theta, 0 \rangle$

$\vec{S}_\varphi = \langle \cos(\varphi)\cos(\theta), \cos(\varphi)\sin\theta), -\sin(\varphi) \rangle$

$$\vec{S}_\theta \times \vec{S}_\varphi = \sin(\varphi) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix}$$

VECTOR LAYED OUT HORIZONTALLY

$$\sin\theta \begin{pmatrix} -\sin\varphi\cos\theta \\ -(\sin(\varphi)\sin(\theta)) \\ -\cos(\varphi)\sin\theta^2 - \cos\varphi\cos^2(\theta) \end{pmatrix}$$

$= \sin\theta\langle -\sin\varphi\cos\theta, -\sin\varphi\sin\theta, -\cos\varphi \rangle$

We popped $\sin(\varphi)$ out of magnitude as $|\sin\varphi|$ is positive on our domain

$$\therefore \iint_S x^2\,ds = \iint_D \sin^3(\varphi)\cos^2(\theta)\,\left|\langle -\sin(\varphi)\cos\theta, -\sin(\varphi)\sin(\theta), -\cos\varphi\rangle\right| dA$$

$$= \iint_S x^2\,ds = \iint \sin^3(\varphi)\cos^2(\theta)\underbrace{\sqrt{\sin\varphi^2\cos\theta^2 + \sin\varphi^2\sin\theta^2 + \cos\varphi^2}}_{1}\,dA$$

$$= \iint \sin^3(\varphi)\cos^2(\theta)\,dA$$

$$\int_0^{2\pi}\cos^2(\theta)\int_0^{\pi}\sin^3(\varphi)\,d\varphi\,d\theta$$

Evaluating inner integral

$$\int_0^{\pi}\sin(\varphi)(1-\cos^2(\varphi))\,d\varphi$$

$u = \cos\varphi$

$du = -\sin(\varphi)\,d\varphi$

$u(\pi) = -1$

$u(0) = 1$

$$\int_1^{-1} -(1-u^2)\,du$$

$$-\left(u - \frac{1}{3}u^3\right)\Big|_1^{-1}$$

$$-1+\frac{1}{3} -1+\frac{1}{3}$$

$\frac{1}{2}$

$$-\left(\frac{-4}{3}\right) = \frac{4}{3}$$

$$\frac{-4}{3}\int_0^{2\pi}\cos^2(\theta)\,d\theta$$

$v = 2\theta$

$dv = 2\,d\theta$

$v(2\pi) = 4\pi$

$v(0) = 0$

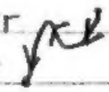
$$\left(\frac{-4}{3}\right)\frac{1}{2}\int_0^{2\pi}(1+\cos(2\theta))\,d\theta = \frac{-2}{3}\int_0^{4\pi} 1+\cos(v)\,dv = \frac{-2}{3}(2\pi+\sin(2\pi)) = \boxed{\frac{-4}{3}\pi}$$

Goal: Build a theory of Surface integrals for vector fields (Analogous to line integrals)

But: We need to think of "orientation" for surfaces first

For Line Integrals:

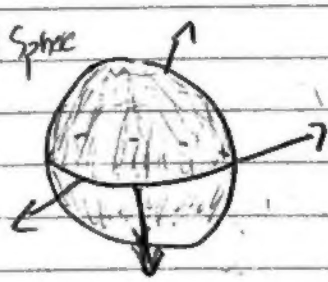
Going LEFT vs RIGHT



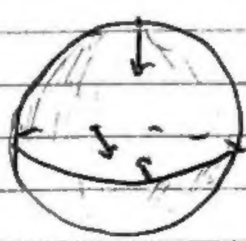
Instead of thinking about orientation, left v right, think about which way the tangent line points.

Orientation for surfaces means a consistent choice of normal to the tangent

Ex: For A Sphere

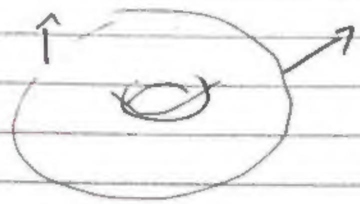


OUTWARD MODEL
OUTWARD ORIENTATION



INWARD ORIENTATION

Alternatively for surfaces that are not closed think "Counter clockwise orientation frame above for the tangent plane"



Positive outward orientation

Can we choose a consistent orientation for every surface?

<u>NO</u>

Möbius Strip → has one side, <u>non</u> orientable

Because of such surfaces our theory doesn't work for non-orientable surfaces. From here on out we will work with orientable surfaces

Note that if/when we choose a parameterization of a surface we automatically chose an orientation. By choosing $\vec{S}(u,v)$ we get n normal:

$$\vec{n}(u,v) = \frac{\vec{S_u} \times \vec{S_v}}{|\vec{S_u} \times \vec{S_v}|}$$

By swapping parameters we swap orientation

Defn: The <u>Flux</u> of v.f. $\vec{v}$ across surface S is $\iint_S \vec{v}\cdot d\vec{S} = \iint_S \vec{v}\cdot\vec{n}\,dS$

PARAMETERIZE → $= \iint_D \vec{v}(u,v)\cdot \underbrace{\frac{\vec{S_u}\times\vec{S_v}}{|\vec{S_u}\times\vec{S_v}|}}_{\text{SCALAR}}\,\underbrace{|\vec{S_u}\times\vec{S_v}|}_{\text{SCALAR}}\,dA$

$$= \iint_D \vec{v}(u,v)\cdot(\vec{S_u}\times\vec{S_v})\,dA$$

where $\vec{S}(u,v)$ is the parameterization of S on domain D

Ex Compute the flux of $\vec{v}(z,y,x)$ across sphere of radius 1 centered at origin

<u>Convention:</u> If orientation is not given, use outward / positive orientation

Sol: As before $\vec{S}(\theta,\varphi) = \begin{pmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{pmatrix}$

$\vec{S}_\theta \times \vec{S}_\varphi = -\sin(\varphi)\begin{pmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{pmatrix}$ * (CALCULATED BEFORE)

Is this positive orientation?

At $P=(1,0,0)$ is $\vec{n}$ going out or in, or $S=(0,\pi/2)$ $\vec{n} = -\sin(\frac{\pi}{2})\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix}$ → Pointing in.

Because we found our parameterization to have negative orientation we need to negate our final result

$$\vec{V}(\theta, \varphi)(\vec{S}_\theta \times \vec{S}_\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \sin\theta \\ \sin\varphi\cos\theta \end{pmatrix} \cdot \begin{pmatrix} \sin\varphi\cos\theta \\ \sin\varphi \sin\theta \\ \cos\varphi \end{pmatrix} (\sin\varphi)$$

$$= \begin{pmatrix} (\sin\varphi \cos\varphi \cos\theta) + \\ (\sin\varphi^2 \sin\theta^2) + \\ (\sin\varphi\cos\varphi\cos\theta) \end{pmatrix} -\sin\varphi$$

$$= -\sin\varphi \left(2\cos(\varphi)\sin(\varphi)\cos\theta + \sin^2(\varphi)\sin^2(\theta)\right)$$

Negate this because of orientation

$$= \iint_D \sin\varphi\left(2\cos(\varphi)\sin(\varphi)\cos\theta + \sin^2\varphi \sin^2(\theta)\right)$$

$$= \underbrace{2\iint_D \sin\varphi^2 \cos\varphi\cos\theta \, dA}_{\int_{\varphi=0}^{\pi} \cos(\varphi)\sin^2(\varphi)\underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{\text{goes to zero}} d\varphi} + \iint \sin^3\varphi \sin^2\theta \, dA$$

$$0 + \iint \sin^3\varphi \sin^2\theta \, dA$$